Perfect graphs: a survey

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Abstract

Perfect graphs were defined by Claude Berge in the 1960s. They are important objects for graph theory, linear programming and combinatorial optimization. Claude Berge made a conjecture about them, that was proved by Chudnovsky, Robertson, Seymour and Thomas in 2002, and is now called the strong perfect graph theorem. This is a survey about perfect graphs, mostly focused on the strong perfect graph theorem.
Key words: Berge graph, perfect graph, graph class, strong perfect graph theorem, induced subgraph, graph colouring.
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1 Introduction

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colours needed to assign a colour to each vertex of G in such a way that adjacent vertices receive different colours. The *clique number* of G, denoted by $\omega(G)$ is the maximum number of pairwise adjacent vertices in G. Every graph G clearly satisfies $\chi(G) \geq \omega(G)$, because the vertices of a clique must receive different colours. A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. A chordless cycle of length $2k+1$, $k \geq 2$, satisfies $3 = \chi > \omega = 2$, and its complement satisfies $k+1 = \chi > \omega = k$. These graphs are therefore *imperfect*. Since perfect graphs are closed under taking induced subgraphs, they must be defined by excluding a familly $\mathcal{F}$ of graphs as induced subgraphs. The *strong perfect graph theorem* (*SPGT* for short) states that the two examples that we just gave are the only members of $\mathcal{F}$. Let us make this more formal.

A *hole* in a graph G is an induced subgraph of G isomorphic to a cycle chordless cycle of length at least 4. An *antihole* is an induced subgraph H of G, such that $\overline{H}$ is hole of $\overline{G}$. A hole (resp. an antihole) is *odd* or *even* according to the number of its vertices (that is equal to the number of its edges). A graph is *Berge* if it does not contain an odd hole nor an odd antihole. The following, known as the *SPGT*, was conjectured by Berge [5] in the 1960s and was the object of much research until it was finally proved in 2002 by Chudnovsky, Robertson, Seymour and Thomas [20] (since then, a shorter proof was discovered by Chudnovsky and Seymour [24]).

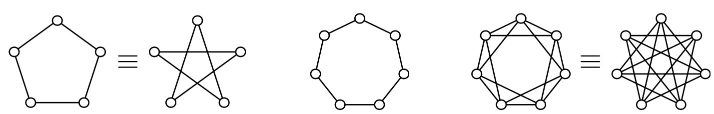

Figure 1: Odd holes and antiholes: $C_5 \equiv \overline{C_5}$, C_7 and $\overline{C_7}$

Theorem 1.1 (Chudnovsky, Robertson, Seymour and Thomas 2002) *A graph is perfect if and only if it is Berge.*

One direction is easy: every perfect graph is Berge, since as we observed already odd holes and antiholes satisfy $\chi = \omega + 1$. The proof of the converse is very long and relies on structural graph theory. The main step is a *decomposition theorem* (Theorem 6.1), stating that every Berge graph is either in a well-understood *basic* class of perfect graphs (see Section 3), or has some *decomposition* (see Section 4). The goal of this survey is to make

this result and its meaning understandable to a non-specialized mathematician. To this purpose, not only the proof is surveyed, but also results that were seminal to it, and some that were proved after it. Our goal is also to present perfect graphs as a lively subject for researchers, by mentioning several attractive open questions at the end of the coming sections. Putting forward up to date open questions is useful, because the proof of the SPGT somehow took back the motivation for many questions, in particular about classes of perfect graphs whose study was supposed to give insight about the big open question.

Let us start now with an open question that obviously has the same flavour as the SPGT. Gyárfás [52] proposed the following generalization of perfect graphs. A graph is *χ-bounded by a function f* if every induced subgraph H of G satisfies $\chi(H) \leq f(\omega(H))$. Hence, perfect graphs are χ-bounded by the identity function. There might exist a short proof that for some function f (possibly fast increasing), all Berge graphs are χ-bounded by f, but so far, the proof of the SPGT is the only known proof of this fact. In [52] the following conjecture is stated. It is still open, but several attempts led to beautiful results, such as Scott [89] and Chudnovsky, Robertson, Seymour and Thomas [21].

Conjecture 1.2 (Gyárfás 1987) *There exists a function f such that for all graphs G, if G has no odd hole, then G is χ-bounded by f.*

Terminology

By the SPGT, perfect graphs and Berge graphs form the same class. Still, we use *Berge* when we mostly rely on excluding holes and antiholes, and *perfect* when we mostly rely on colourings. Anyway, this distinction has to be kept when we sketch the proof of the SPGT. A *minimally imperfect graph* is an imperfect graph every proper induced subgraph of which is perfect. A restatement of the SPGT is that minimally imperfect graphs are precisely the odd holes and antiholes. Many statements about minimal imperfect graphs are therefore trivial to check by using the SPGT, but when proving the SPGT, it is essential to prove them by other means. Observe that a minimal (or minimum) counter-example to the SPGT has to be a Berge minimally imperfect graph.

We mostly follow the terminology from [20] that is sometimes not fully standard. Since all this theory deals with induced subgraphs, we write G *contains* H to mean that H is an induced subgraph of G. We simply write *path* instead of chordless path or induced path. When a and b are vertices

of a path P, we denote by aPb the subpath of P whose ends are a and b. A subset A of $V(G)$ is *complete* to a subset B of $V(G)$ if A and B are disjoint and every vertex of A is adjacent to every vertex of B (we also say that B is A-*complete*).

We use the prefix *anti* to mean a property or a structure of the complement (like in holes and antiholes). For instance, an *antipath* in G is a path in $\overline{G}$; A is *anticomplete* to B means that no edge of G has an end in A and the other one in B; a graph G is *anticonnected* if its complement is connected. By $C(A)$ we mean the set of all A-complete vertices in G and by $\overline{C}(A)$ the set of all A-anticomplete vertices.

By *colouring* of a graph, we mean an optimal colouring of the vertices.

Further reading

Since we focus on the SPGT and its proof, most of this survey is on the structure of Berge graphs, and we omit many other important (more important?) aspects of the theory of perfect graphs, such as the ellipsoïd method used by Gröstchel, Lovász and Schrijver [51] to colour in polynomial time any input perfect graph. The theory of semi-definite programming started there.

Several previous surveys exist (and we try not to overlap them too much). As far as we know, all of them were written before of just after the proof of the SPGT. The survey of Lovász [69] from the 1980s is still a good reading. Two books are completely devoted to perfect graphs [7, 81], and contain a lot of material that will be cited in what follows. Part VI of the treatise of Schrijver [88] is a good survey on perfect graphs (it is the most comprehensive). Chudnovsky, Robertson, Seymour and Thomas [19] wrote a good survey just after their proof. Roussel, Rusu and Thuillier [85] wrote a good survey about the long sequence of results, attempts and conjectures that finally led to the successful approach.

About more historical aspects, see Section 67.4g in Schrijver [88]. Berge and Ramírez Alfonsín [6] wrote an article about the origin of perfect graphs. Seymour [90] wrote an article, that is interesting even to a non-mathematician, telling the story of how the SPGT was proved.

2 Lovász's perfect graph theorem

As pointed out by Preissmann and Sebő [79], the following conveniently gives a weak (ii) and a strong (iii) characterization of perfect graphs. Hence, to prove perfection, checking the weak one is enough, while to use perfection,

one may rely on the strong one. This allows a kind of leverage that is used a lot in the sequel.

Lemma 2.1 (Folklore) *Let G be a graph. The three statements below are equivalent.*

(i) *G is perfect.*

(ii) *For every induced subgraph H of G and every $v \in V(H)$, H contains a stable set that contains v and intersects all maximum cliques of H.*

(iii) *For every induced subgraph H of G, H contains a stable set that intersects all maximum cliques of H.*

Proof. To prove that (i) implies (ii), consider an optimal colouring of H and the colour class S that contains v. So S is a stable set, and it must intersect all maximum cliques of H for otherwise $\chi(H \setminus S) \geq \omega(H)$, a contradiction. Trivially, (ii) implies (iii). To prove that (iii) implies (i), consider the following greedy colouring algorithm: (step 1) set $i \leftarrow 1$ and $H \leftarrow G$; (step 2) while H is non-empty, consider a stable set of H as in (iii), give it colour i, and set $H \leftarrow H \setminus S$, $i \leftarrow i+1$. Since ω decreases at each step, this algorithm produces a colouring of G with $\omega(G)$ colours, and it can be processed for any induced subgraph of G. Thus, G is perfect. □

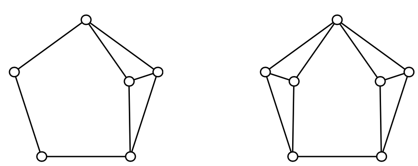

Figure 2: Replication: G and G'

Replicating a vertex v of a graph G means adding a new vertex v' adjacent to v and all neighbors of v. As an example, consider the (non-perfect) graph G obtained by replicating one vertex of C_5. Clearly, $\chi(G) = \omega(G) = 3$ (see Fig. 2). However, by replicating any vertex of degree 2 in G, a graph G' such that $\chi(G') = 4 > \omega(G') = 3$ is obtained. This shows that the property $\chi = \omega$ is not preserved by replication. Therefore, the following lemma due to Lovász [68, 70] and known as the *replication lemma* is more surprising than it may look at first glance.

Lemma 2.2 (Lovász 1972) *Perfection is preserved by replication.*

Proof. Let G be a perfect graph and G' a graph obtained from G by replicating a vertex v. We show that G' satisfies the characterization (iii) from Lemma 2.1. Let H be an induced subgraph of G'. We look for a stable set S that intersects all maximum cliques of H. If H contains at most one vertex from $\{v, v'\}$, then it is isomorphic to a an induced subgraph of G, so clearly S exists. Otherwise, $H \setminus v'$ is perfect, so, by charactization (ii), there exists a stable set S that contains v and intersects all maximum cliques of $H \setminus v'$. In fact S intersects all maximum cliques of H, because a maximum clique of H contains v if and only if it contains v'. $\square$

The following was conjectured by Berge as the *weak perfect graph conjecture* and is now called the *perfect graph theorem.* We give the proof that, among the available ones, we feel the most related to the rest of this survey. The *stable set number* of a graph G, denoted by $\alpha(G)$, is the maximum number of pairwise non-adjacent vertices. Observe that for all graphs G, $\alpha(G) = \omega(\overline{G})$.

Theorem 2.3 (Lovász 1972) *If a graph is perfect, then its complement is perfect.*

Proof. Let G be a perfect graph. Construct G' as follows: start from G and replicate $\alpha_v - 1$ times every vertex v, where α_v is the number of maximum stables sets of G that contain v. Note that replicating -1 time means deleting the vertex, and replicating 0 times means doing nothing. From its construction, G' can be covered by k disjoint maximum stable sets, that therefore form an optimal colouring of G'. Since G' is perfect by Lemma 2.2, if follows that G' has a clique K' of size k. Since a clique and a stable set intersect in at most one vertex, K' intersect all maximum stable sets of G'. Now construct a clique K of G, by taking for each vertex of K' the vertex of G it is replicated from. The clique K that we obtain intersects all maximum stable sets of G. By the same lines, a clique intersecting all maximum stable sets can be found in any induced subgraph of G. Hence the complement of G satisfies condition (iii) of Lemma 2.1 and is therefore perfect. $\square$

In [52], the question of generalizing Theorem 2.3 is discussed. Since [52], hardly any progress occurred in this direction. In particular the following neat generalization of Theorem 2.3 is still open.

Conjecture 2.4 (Gyárfás 1987) *There exists a function f such that for all graphs G, if G is χ-bounded by $x \mapsto x + 1$, then its complement is χ-bounded by f.*

Further reading

The perfect graph theorem has a polyhedral proof found by Fulkerson [46] related to polyhedral characterizations of perfect graphs discovered by Fulkerson [46] and Chvátal [29]. Lovász [67] proved a deep characterization of perfect graphs suggested by Hajnal: a graph G is perfect if and only if every induced subgraph H of G satisfies $\alpha(H)\omega(H) \geq |V(H)|$ (this can be proved by a simple argument relying on linear algebra discovered by Gasparian [49], see also [40, 10], and Section 12 below for applications of linear algebraic methods to perfect graphs). This characterization implies that deciding the perfection of an input graph is a CoNP problem (see [79] for more about that). Since the characterization is self-complementary, it gives another proof of the perfect graph theorem. This characterization is the starting point of many developments of great signifiance, such as the theory of partitionable graphs (see the survey of Preissmann and Sebő [79]). It has deep connections with combinatorial optimization as explained in a book of Cornuéjols [36].

3 Basic graphs

In this section, we survey the five basic classes that are used in the proof of the SPGT. We denote by $\theta(G)$ the chromatic number of $\overline{G}$, by $\nu(G)$ the maximum size of a matching in G, by $\Delta(G)$ the maximum degree of a vertex in G, by $\tau(G)$ the minimum number of edges of G needed to cover all vertices of G, and by $\chi'(G)$ the minimum number of colours needed to assign a colour to each *edge* of G in such a way that adjacent edges receive different colours. Bipartite graphs are easily checked to be perfect. So, by Theorem 2.3, their complements are also perfect, which can be restated as 'every bipartite graph G satisfies $\theta(G) = \alpha(G)$'. Since for any triangle-free graph G, $|V(G)| = \theta(G) + \nu(G) = \alpha(G) + \tau(G)$, we obtain that every bipartite graph G satisfies $\nu(G) = \tau(G)$. This can be rephrased as: 'the complements of line graphs of bipartite graphs are perfect'. By applying again Theorem 2.3, we obtain that line graphs of bipartite graphs are perfect, which can be restated as 'every bipartite graph G satisfies $\Delta(G) = \chi'(G)$'. Hence, Theorem 2.3 implies the perfection of three among the four *historical basic* classes of perfect graphs: bipartite graphs, their complements, their line graphs, the complements of their line graphs. Interestingly, this was all proved directly by Kőnig [60, 59], long before the definition of perfect graphs.

We now turn our attention to a less classical class that is first presented

in the proof of the SPGT: *double split graphs*. As it is presented in [20], the class is not closed under taking induced subgraphs, which is sometimes not convenient. So we prefer here to define directly *doubled graphs*, that are easily seen to form the class of induced subgraphs of double split graphs (as defined in [20]).

A *good partition* of a graph G is a partition (X, Y) of $V(G)$ (possibly, $X = \emptyset$ or $Y = \emptyset$) such that:

- Every component of $G[X]$ has at most two vertices, and every anticomponent of $G[Y]$ has at most two vertices.
- For every component C_X of $G[X]$, every anticomponent C_Y of $G[Y]$, and every vertex v in $C_X \cup C_Y$, there exists at most one edge and at most one antiedge between C_X and C_Y that is incident to v.

A graph is *doubled* if it has a good partition (for the sake of completeness, let us mention that a *double split graph* is a doubled graph such that $G[X]$ (resp. $G[Y]$) has at least two components (resp. anticomponents) and all components (resp. anticomponents) of $G[X]$ (resp. $G[Y]$) have two vertices). Doubled graphs are easily seen to be perfect by a direct colouring argument. They are closed under taking induced subgraphs and complements. A graph is *basic* if it belongs to at least one of the five classes defined here: bipartite, complement of bipartite, line graph of bipartite, complement of line graph of bipartite, and doubled graphs. For each basic class, the characterization by excluding induced subgraphs is known (see Beineke [4] for line graphs and Alexeev, Fradkin and Kim [3] for doubled graphs); the recognition can be performed in polynomial time (see Lehot [61] or Roussopoulos [86] for line graphs and Chudnovsky, Trotignon, Trunck and Vušković [28] for doubled graphs). Also the colouring and maximum clique problems can be solved in polynomial time (see Schrijver [88] for the 'historical classes' and [28] for doubled graphs). However, doubled graphs are so simple that methods faster than those in [28] should exist.

Question 3.1 *Is there a linear time algorithm for the recognition, the colouring and the maximum clique of a doubled graph?*

Further reading

In this section, we focused on the basic graphs that play an important role in the proof of the SPGT. But any class of graphs whose perfection is simple to prove can potentially serve as a basic class of a decomposition theorem,

so all classes are potentially of interest. The book of Brandstädt, Le and Spinrad [11] is on general graph classes, but contains a lot of material on perfect graphs. The most complete catalog of classes of perfect graphs seems to be written by Hougardy [57], that describes 120 classes. Also Chapter 66 in Schrijver [88] contains a very complete survey about classes of perfect graphs. The book by Golumbic [50] surveys algorithmic aspects of several classes of perfect graphs.

Perhaps the most important class that we omit to present here is the seminal class of hole-free graphs, known as *chordal* graphs and introduced by Dirac [41] and Gallai [47]. It is the first class with a decomposition theorem, has some connections with the graph minors theory and tree-width, and is also important in fast graph searching algorithms. About that, a good starting point is Sections 9.7–9.8 in the book of Bondy and Murty [10]. Another important class, that has connections with ordered sets, is the class of comparability graphs introduced by Gallai [48] (the English translation by Maffray and Preissmann [71] contains a short survey).

4 Decompositions

By *decomposition* of a graph we mean a way to partition its vertices with some prescribed adjacencies. A decomposition is *useful* if it can be proved that a minimum counter-example to the SPGT cannot admit the decomposition. Indeed, suppose that we can prove a statement such as: 'every Berge graph is either basic or has a useful decomposition'. The SPGT can then be proved as follows: consider a minimum counter-example, i.e. a Berge graph, imperfect, and of minimum size. It does not admit the decomposition (because the decomposition is useful), and since it is imperfect, it cannot be basic, a contradiction to the statement.

The simplest useful decomposition, first observed in this context by Gallai [47], is the *clique cutset*, that is a clique whose removal yields a disconnected graph. It is easily seen to be useful, because gluing two perfect graphs along a clique yields a perfect graph.

In a graph G, *substituting* a graph H for a vertex v, means deleting v, adding a copy of H, making every neighbor of v complete to H, and every non-neighbor of v anticomplete to H. Along the lines of the proof of Lemma 2.2, it is easy to prove that substituting a perfect graph for a vertex of a perfect graph yields a perfect graph (this is therefore a variant of Lovász's replication lemma). It follows easily that a minimally imperfect graph does not admit a homogeneous set, where a *homogeneous set* of a

graph G is a set $H \subseteq V(G)$ such that $1 < |H| < |V(G)|$ and every vertex of $G \setminus H$ is either complete or anticomplete to H.

A *1-join* of a graph G, first defined by Cunningham [38], is a partition (X, Y) of $V(G)$ such that $|X| \geq 2$, $|Y| \geq 2$, and there exist $A \subseteq X$ and $B \subseteq Y$ such that A is complete to B and no other edges exist from X to Y. Again, 1-joins can be proved to be useful (this is proved by Cunningham [38], Bixby [9] and it also follows from Lemma 4.1 below).

Note that clique cutsets, homogeneous sets and 1-joins do not appear explicitly in Theorem 6.1. This is because they are not formally necessary, since their presence implies that the graph is basic or has another decomposition (namely the balanced skew partition, to be defined soon). We mention them because they are somehow present 'implicitly'; this sometimes shows up naturally in attempts to use Theorem 6.1 for algorithmic purpose. We now turn our attention to the decompositions that are actually used in Theorem 6.1. For each definition, we use the definition from [20], that sometimes differ slightly from the definition given in the paper where the decomposition is first presented (and where the proof of the usefulness of the decomposition is given).

A *2-join* of a graph G, first defined by Cornuéjols and Cunningham [37], is a partition (X_1, X_2) of $V(G)$ such that there exist disjoint non-empty sets $A_1, B_1 \subseteq X_1$, $A_2, B_2 \subseteq X_2$ satisfying:

- A_1 is complete to A_2, B_1 is complete to B_2 and these edges are the only ones between X_1 and X_2;
- $|X_i| \geq 3$, $i = 1, 2$;
- every component of $G[X_i]$ intersects A_i and B_i, $i = 1, 2$; and
- if $|A_i| = |B_i| = 1$, then $G[X_i]$ is not a path of length two joining the members of A_i and B_i, $i = 1, 2$.

Cornuéjols and Cunningham [37] proved that a minimally imperfect graph admiting a 2-join must be an odd hole (so, 2-joins are useful). A reader who pays attention to technicalities may notice that here, *path 2-joins* are allowed (these are 2-joins such that for some $i \in \{1, 2\}$, $|A_i| = |B_i| = 1$ and $G[X_i]$ is a path from the unique vertex in A_i to the unique vertex in B_i). Some papers (mostly, these cosigned by Conforti, Cornuéjols or Vušković) restrict the notion of 2-joins to *non-path 2-joins*. In Section 10, the relevance of excluding path 2-joins is discussed. When X_i, A_i and B_i are as above, it is customary to set $C_i = X_i \setminus (A_i \cup B_i)$. It is easy to prove that in Berge

graphs, all paths from A_i to B_i with interior in C_i have the same parity (otherwise, an odd hole exists). Therefore, there are two kinds of 2-joins, according to this parity: *odd* and *even* 2-joins. If (X_1, X_2) is a 2-join of $\overline{G}$, then it is a *complement 2-join* of G.

When G is a graph and $A \subseteq V(G)$, we denote by $C(A)$ the sets of vertices of G complete to A and by $\overline{C}(A)$ the set of vertices of G anticomplete to A. A *homogeneous pair* (first defined in a slightly different way by Chvátal and Sbihi [31] who proved that they are useful) is a pair of disjoint sets $A, B \subseteq V(G)$ such that $|A|, |B| \geq 2$, every vertex of A has a neighbor and a non-neighbor in B, every vertex of B has a neighbor and a non-neighbor in A, and A, B, $C(A) \cap \overline{C}(B)$, $\overline{C}(A) \cap C(B)$, $C(A) \cap C(B)$, $\overline{C}(A) \cap \overline{C}(B)$ are all non-empty and partition $V(G)$.

All the decompositions presented so far are nice in the following sense. When applied recursively, they yield decomposition trees of polynomial size that allow solving several problems. The machinery is too heavy to be presented here, see Section 12. We now turn our attention to other kinds of cutset that do not have this nice property.

A *star cutset* (first defined by Chvátal [30]) in a graph G is a set S of vertices such that $G \setminus S$ is disconnected and S contains a vertex v, called the *center*, complete to $S \setminus v$. The following is known as the *star cutset lemma* [30]. It remarkably generalizes the usefulness of clique cutsets, 1-joins and homogeneous sets (because these decompositions all imply the presence of a star cutset).

Lemma 4.1 (Chvátal 1985) *A minimally imperfect graph has no star cutset.*

Proof. Let G be a minimally imperfect graph and suppose for a contradiction that G has a star cutset S centered at v. Let (X, Y) be a partition of $V(G \setminus S)$ such that $|X|, |Y| \geq 1$ and X is anticomplete to Y. We now prove that G satisfies the condition (iii) from Lemma 2.1 (this implies that G is perfect, giving the contradiction). Since every proper induced subgraph of G is perfect, it just remains to find the desired stable set in G. By condition (ii) from Lemma 2.1, there exists a stable set A_X in $G[S \cup X]$ that contains v and intersects all maximum cliques of $G[S \cup X]$. A similar stable set A_Y exists in $G[S \cup Y]$. Now, $A_X \cup A_Y$ is a stable set of G that intersects all maximum cliques of G. $\square$

It is quite easy to turn the proof above into a colouring algorithm (that would for instance colour any perfect graph every induced subgraph of which

is either basic or decomposable by a star cutset). The algorithm would output a stable set that intersects all maximum cliques, and along the lines of the proof of Lemma 2.1, this gives a colouring algorithm. Unfortunately, this algorithm does not run in polynomial time, because the star cutset can be very big, for instance be the entire graph except two vertices. In this case, the complexity analysis of the recursive calls leads to an exponential number of calls. There is a similar problem with the generalizations that we consider now. This is the main reason why the decomposition of Berge graphs does not lead to a polynomial time colouring algorithm.

A *skew partition* (first defined by Chvátal [30]) of a graph G is a partition (A, B) of $V(G)$ such that $G[A]$ is not connected, and $G[B]$ is not anticonnected. In this case, we say that B is a *skew cutset.* Following a prophetic insight that some *self-complementary* decomposition generalizing the star cutset should play some role, Chvátal conjectured that a minimally imperfect graph has no skew partition, and a less formal statement, that skew partitions should appear in the decomposition of Berge graphs. Observe that if a graph on at least 5 vertices with at least one edge has a star cutset, then it has a skew partition. The proof of Chv́atal's conjectures escaped the researchers, but several fruitful attempts were made. In particular, many special kinds of skew partitions were proved not to be in minimal imperfect graphs (see Reed [83] for a survey). In the opposite direction, a generalization of skew partitions was proved to decompose all Berge graphs in the following theorem [34]. A *double star cutset* in a graph G is a set $S \subseteq V(G)$ such that $G \setminus S$ is disconnected and S contains an edge uv such that every vertex of S is adjacent to at least one of u, v. Note that for a Berge graph G, the following gives in fact two pieces of information: one for G, one for $\overline{G}$.

Theorem 4.2 (Conforti, Cornuéjols and Vušković 2004) *A graph with no odd hole is either basic, or has a 2-join or a double star cutset.*

One of the breakthroughs made in the proof of the SPGT is the concept of *balanced skew partition.* For a graph G, a partition (skew or not) (A, B) of $V(G)$ is *balanced* if every path of length at least 3, with ends in B and interior in A, and every antipath of length at least 3, with ends in A and interior in B has even length. It is straightforward to check that a partition (A, B) of a Berge graph is balanced if and only if adding a vertex complete to B and anticomplete to A yields a Berge graph.

As we will see, the notion of balanced skew partition is sufficiently particular to allow a short proof that a minimum counter-examples to the SPGT

cannot contain it, and sufficiently general to be found in all non-basic Berge graphs that cannot be decomposed otherwise. Interestingly, Zambelli [97] notices that if a Berge graph on at least five vertices and with at least one edge has a star cutstet, then it has a balanced skew partition.

Lemma 4.3 *If (A, B) is a balanced partition of a perfect graph G, and if every Berge graph of size at most $|V(G)|+1$ is perfect, then $G[B]$ admits a colouring that can be extended to a colouring of G.*

Proof. Consider the graph G' obtained by adding a clique of size $k = \omega(G) - \omega(G[B])$ complete to B and anticomplete to A. It is Berge because (A, B) is balanced. So it is perfect when $k \leq 1$ from our assumption, and it is also perfect and when $k \geq 2$ by several applications of Lemma 2.2. Observe that $\omega(G') = \omega(G)$. An $\omega(G)$ colouring of G' yields a colouring of $G[B]$ that extends to a colouring of G. □

Theorem 4.4 (Chudnovsky, Robertson, Seymour and Thomas 2002) *A minimum imperfect Berge graph admits no balanced skew partition.*

Proof. Let G be minimum imperfect Berge graph. Hence, $\chi(G) > \omega(G)$. Note that by Theorem 2.3, $\overline{G}$ is also a minimum imperfect Berge graph. Let (A, B) be a balanced skew partition in G. So, A partitions into two sets A_1 and A_2 anticomplete to one another, and B partitions into two sets X and Y complete to one another. By Lemma 4.1, $|A_1|, |A_2| \geq 2$, for otherwise, the unique vertex in A_1 or A_2 would be the center of a star cutset in $\overline{G}$. From the minimality of G, it follows that every Berge graph of size $|V(G[B \cup A_i])| + 1$ is perfect, $i = 1, 2$. By Lemma 4.3, consider an $\omega(G[B])$ colouring C_i of $G[B]$ that extends to a colouring of $G[B \cup A_i]$. Let X_i be the set of vertices of $G[B \cup A_i]$ whose colour in the colouring C_i is present in X and let $Y_i = (B \cup A_i) \setminus X_i$. Because of the colouring C_i, $\omega(G[X_i]) = \omega(G[X])$. So $\omega(G[X_1 \cup X_2]) = \omega(G[X])$. By the minimality of G, it follows that $G[X_1 \cup X_2]$ has an $\omega(G[X])$-colouring. Because of the colouring C_i, $\omega(G[Y_i]) = \omega(G) - \omega(G[X_i]) = \omega(G) - \omega(G[X])$. So, $\omega(G[Y_1 \cup Y_2] = \omega(G) - \omega(G[X])$. By the minimality of G, it follows that $G[Y_1 \cup Y_2]$ has an $(\omega(G) - \omega(G[X]))$-colouring. It follows that G has an $\omega(G)$-colouring, a contradiction. □

The following is maybe hopeless, because a proof would imply a direct argument for the skew partition conjecture (no such argument exists today). Also a proof of the following together with Theorem 4.2 would yield a new

proof the SPGT. Observe that antiholes of length at least 6 have double star cutsets.

Question 4.5 *Find a direct proof of the following: if G is a minimum Berge imperfect graph, then at least one of $G, \overline{G}$ admits no double star cutset.*

Further reading

Rusu [87] wrote a survey about cutsets in perfect graphs, see also [85]. Reed [83] wrote a survey about skew partitions (on which this section is mostly based). It shows that many ideas of the proof presented above for balanced skew partitions are implicitly contained in several papers, namely in Hoàng [55], Olariu [76] and Roussel and Rubio [84].

An important question about decompositions is their detection in polynomial time. The following decompositions can all be detected in polynomial time: clique cutset (in time $O(nm)$, Tarjan [91]), homogeneous set (in time $O(n + m)$, see Habib and Paul [54]), 1-join (in time $O(n + m)$, see Charbit, de Montgolfier and Raffinot [77]), 2-join (in time $O(n^2m)$, Charbit, Habib, Trotignon and Vušković [13]), homogeneous pair (in time $O(n^2m)$, Habib, Mamcarz and de Montgolfier [53]), skew partitions (in time $O(n^4m)$, Kennedy and Reed [58]). Trotignon [92] showed that balanced skew partition are NP-hard to detect, but devised an $O(n^9)$-time non-constructive algorithm that certifies whether an input *Berge* graph has or not a balanced skew partition.

5 Truemper configurations

Before going further, we need to define several special kinds of graphs, known as *Truemper configurations.* They appear in many contexts (sometimes in older papers, such as [96]).

A *prism* is a graph made of three vertex-disjoint paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except these of the two triangles. Observe that a prism in a Berge graph must have the lengths of the three paths of the same parity. The prism is *odd* or *even* according to this parity.

A *pyramid* is a graph made of three paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist

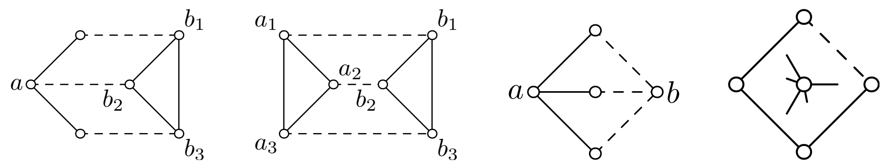

Figure 3: Pyramid, prism, theta and wheel (dashed lines represent paths)

between the paths except these of the triangle and the three edges incident to a.

A *theta* is a graph made of three internally vertex-disjoint paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b.

Observe that the lengths of the paths in the three definitions are designed so that the union of any two of the paths form a hole.

A *wheel* is a graph formed by a hole H together with a vertex that have at least three neighbors in the hole.

A *Truemper configuration* is a graph isomorphic to a prism, a pyramid, a theta or a wheel. As we will see, Truemper configurations play a special role in the proof of the SPGT. First, a Berge graph has no pyramid (because among the three paths of a pyramid, two have the same parity, and their union forms an odd hole). This little fact is used very often to provide a contradiction when working with Berge graphs. As we will soon see, a long part of the proof is devoted to study the structure of a Berge graph that contains a prism, and another long part is devoted to a Berge graph that contains a wheel. And at the very end of the proof, it is proved that graphs not previously decomposed are bipartite, just as Berge thetas are. Note also that prisms can be defined as line graphs of thetas. This use of Truemper configurations is seemingly something deep and general as suggested by Vušković in a very complete survey [95] about Truemper configurations and how they are used (sometimes implicitly) in many decomposition theorems.

So far, no systematic study of the exclusion of Truemper configurations has been made. The most interesting question is perhaps the following.

Question 5.1 *Is there a decomposition theorem for graph that do not contain wheels as induced subgraphs? Can they be recognized in polynomial time? Are they all χ-bounded by the same function?*

Further reading

About Truemper configurations, the survey of Vušković [95] is the best reading. About excluding wheels, see Aboulker, Radovanović, Trotignon and Vušković, [2]. To see how Truemper configurations appear naturally in the definition of several classes that generalize chordal graphs, see Aboulker, Charbit, Trotignon and Vušković [1].

Testing whether a graph contains or not some type of Truemper configuration is a question of interest. Detecting a theta in some input graph can be done in time $O(n^{11})$ (see Chudnovsky and Seymour [27]) and a pyramid in time $O(n^9)$ (Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [17]). Detecting a prism is NP-complete (Maffray and Trotignon [72]) and detecting a wheel is not known to be polynomial or NP-complete as mentioned already (see [2]). Detecting a prism or a pyramid can be done in time $O(n^5)$ (Maffray and Trotignon [72]). Detecting a theta or a pyramid can be done in time $O(n^7)$ (Maffray, Trotignon and Vušković [74]). Detecting a prism or a theta can be done in time $O(n^{35})$ (Chudnovsky and Kapadia [18]). For similar questions, see Lévêque, Lin, Maffray and Trotignon [63].

6 Strategy of the proof

The main result in [20] is Theorem 6.1 below, and as we know from the previous sections, it implies the SPGT. Its statement is the result of a long sequence of attempts by many researchers, as explained in the introduction of [20] or in [85]. A slight variant on what seems now to be the right statement was first conjectured by Conforti, Cornuéjols and Vušković [33]. They proved it in the square-free case, and some of the arguments that they discovered are essential in the strategy described below (in particular, the attachments to prisms, and the use of Truemper configurations).

Theorem 6.1 (Chudnovsky, Robertson, Seymour and Thomas 2002) *Every Berge graph is basic, or has a 2-join, a complement 2-join, a homogeneous pair or a balanced skew partition.*

The strategy used by Chudnovsky, Robertson, Seymour and Thomas to prove Theorem 6.1 is classical in structural graph theory. It consists in identifying a 'dense' basic class, and a 'sparse' basic class as we explain now. The 'dense' basic class does not contain the obstruction (here an odd hole or antihole) of course, but 'almost' contains it, so that if a graph G contains an induced dense subgraph H, then any vertex exterior to H must

attach in a very specific way to H, either enlarging the basic graph to a bigger basic graph, or entailing a decomposition. Therefore, for the sake of proving the decomposition theorem, it can be assumed that this particular class of basic graphs is excluded. Then the process can be iterated with a new kind of dense basic graphs. The sparse class is what remains when all dense substructures are excluded. Observe that this method is in some sense wiser than proofs by induction. Finding the right induction hypothesis is time consuming because for every failure, one has to restart from scratch, while a lemma stating that some dense substructure entails a decomposition is just a true statement, that can be used forever, even if the strategy of the proof changes.

For proving Theorem 6.1 the sparse class is formed by bipartite graphs and their complements. The dense class is more complicated. At the beginning of the proof, it is formed by 'sufficiently' connected line graphs, their complements and doubled graphs. The simplest line graphs in this context are the odd and even prisms (that are the line graphs of bipartite thetas). To understand why prisms and their generalizations are 'dense', the reader can check as an exercice that a Berge graph formed of a prism and one vertex not in the prism is either a bigger line graph, or has a 2-join, or some kind of skew partition, namely a star cutset. To this purpose, it is very convenient to know that Berge graphs have no pyramid, and what makes this work is that prisms are 'close to' containing pyramids. Typical cases that should pop out from a proof attempt are represented in Fig. 4. Then the reader might try to prove a similar statement for the line graph of the 2-subdivision of K_4, or to prove similar statements with the vertex outside of the prism replaced by some path with neighbors in exactly two paths of the structure for instance. All this should lead to variants on 10.1 from [20], whose proof is easy to read since it does not rely on any technical lemma. More about attachments to prisms is explained in Section 8 below.

A sequence of about a dozen dense basic graphs is considered: first, several kinds of line graphs of bipartite subdivisions of K_4, then even prisms, then long prisms (long means that at least one of the paths has length at least 2), then the *double diamond* (see Fig 5), then various kinds of wheels (that are not basic, but that contain skew partitions), and finally antiholes of length at least 6. For each of these dense basic classes, it is proved that containing it entails being basic or having some decomposition, and therefore, the Berge graphs handled next may be assumed not to contain this kind of induced subgraph. At the end of this process, so many induced subgraphs are excluded that the graph under consideration, or its complement, is bipartite. Needless to say, identifying this long sequence of 'dense'

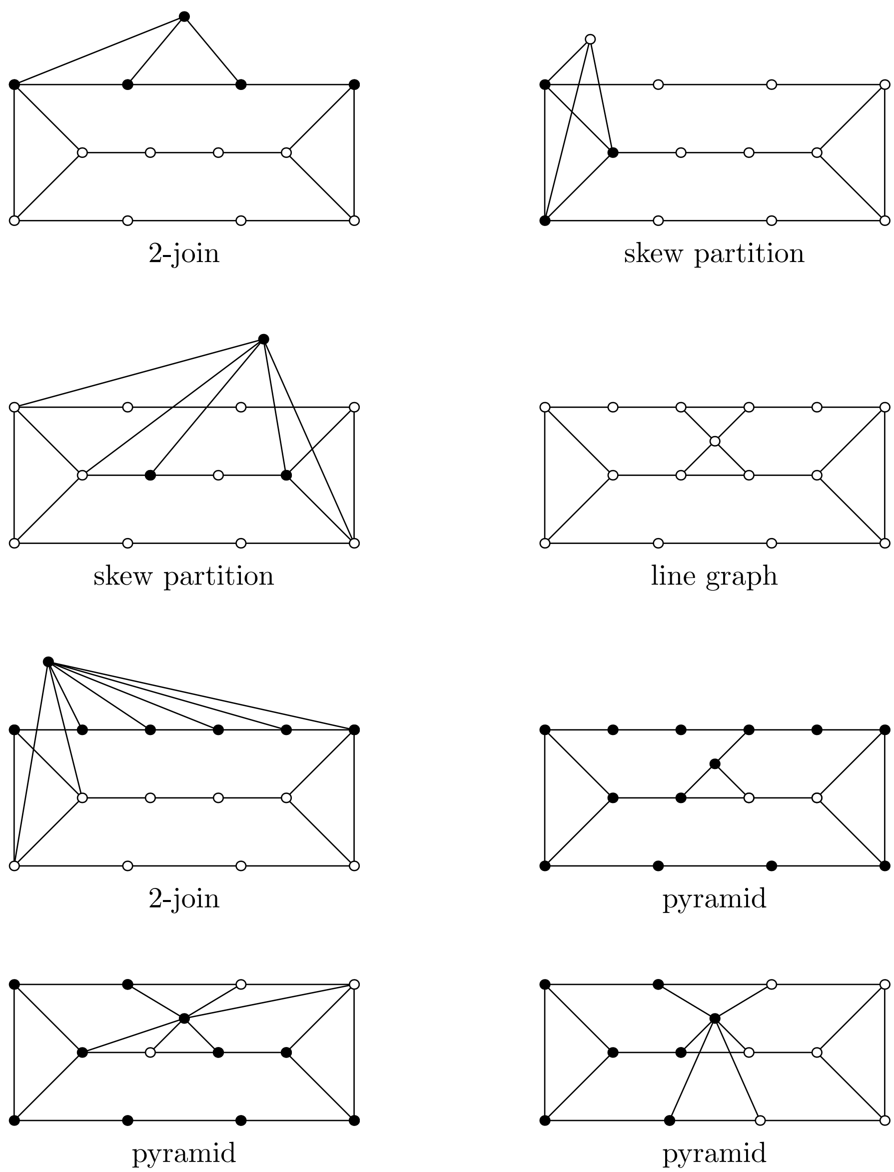

Figure 4: Various ways to attach a vertex to a prism

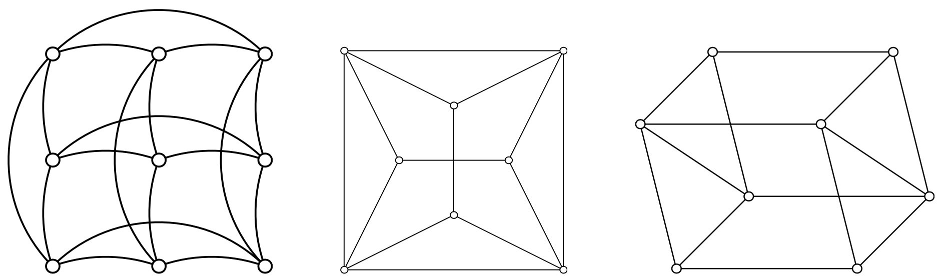

Figure 5: $L(K_{3,3})$, $L(K_{3,3} \setminus e)$ and the double diamond

graphs is *tour de force*, especially since for each of them, the technicalities are really involved. In particular, the self-complementary graphs $L(K_{3,3})$ and $L(K_{3,3} \setminus e)$ (see Fig. 5) are a problem since they are basic in many ways (they are both line graph and complement of line graphs, and the later is also a doubled graph). Therefore, there are several ways to describe their structure, depending on the basic class, and the relevant one is known only from the rest of the graph.

Despite all the deep technicalities, the objects considered in the proof of Theorem 6.1 are very combinatorial. This leads to the following question.

Question 6.2 *Can the proof of Theorem 6.1 be transformed into a polynomial time algorithm whose input is any graph G and whose output is either an odd hole, an odd antihole, or a partition of the vertices of G certifying one of the outcomes of Theorem 6.1?*

Further reading

In [20], the global strategy of the proof is well explained at the beginning. More about the strategy is to be found in [90] and [19]. How structural methods can be used generally for classes closed under taking induced subgraphs is discussed in Chudnovsky and Seymour [23] and in Vušković [95]. To make a start on Question 6.2, the first step is the detection in polynomial time of the structures that are used in the proof (line graphs of a bipartite subdivision of K_4, even prism and odd prism). Apart from wheels (the complexity of detecting them is unknown), they all can be detected in polynomial time in Berge graph, see Maffray and Trotignon [72].

7 The Roussel–Rubio lemma

We are now ready to invistigate some technicalities of the proof of the SPGT. A lemma due to Roussel and Rubio [84] is used at many steps in [20]. In fact, the authors of [20] rediscovered it (in joint work with Thomassen) and initially named it the *wonderful lemma* because of its many applications.

The Roussel–Rubio lemma states that, in a sense, any anticonnected set of vertices of a Berge graph behaves like a single vertex. How does a vertex v 'behave' in a Berge graph? If a path of odd length (at least 3) has both ends adjacent to v, then v must have other neighbors in the path, for otherwise there is an odd hole. The lemma states roughly that an anticonnected set T of vertices behaves similarly: if a path of odd length (at least 3) has both ends complete to T, then at least one internal vertex of the path is also complete to T. In fact, there are two situations where this statement fails (outcomes (iii) and (iv) below), and for the sake of induction, it is convenient to give similar properties for a path of even length. All this results in a more complicated statement. A *T-edge* is an edge whose ends are T-complete. When $P = xx' \dots y'y$ is a path of length at least 3, a *leap* for P is a pair of non-adjacent vertices u, v such that $N(u) \cap V(P) = \{x, x', y\}$ and $N(v) \cap V(P) = \{x, y', y\}$. Observe that if u, v is a leap for P, then $V(P) \cup \{u, v\}$ induces a prism. We denote by P^* the interior of a path P.

Lemma 7.1 (Roussel and Rubio 2001) *Let T be an anticonnected set of vertices in a Berge graph G. If P is a path, vertex-disjoint from T and whose ends are T-complete, then one the following holds.*

(i) P has even length and has an even number of T-edges;

(ii) P has odd length and has an odd number of T-edges;

(iii) P has odd length at least 3 and there is a leap for P in T;

(iv) P has length 3 and its two internal vertices are the endvertices of an antipath of odd length whose interior is in T.

Proof. We prove the lemma by induction on $|V(P) \cup T|$. If P has length at most 2, then we have outcome (i) or (ii). So let us assume that P has length at least 3. Put $P = xx' \cdots y'y$. Let us suppose that outcomes (iii) does not hold for P. We distinguish between three cases.

Case 1: There is a T-complete vertex in P^*. Let z be such a vertex. By induction, we can apply the lemma to the path xPz and T. If xPz has odd length and there is a leap $\{u, v\}$ for xPz in T, then $(xPz)^* \cup \{u, v, y\}$

induces an odd hole. If xPz has length 3 and its two internal vertices are the endvertices of an odd antipath Q whose interior is in T, then $Q \cup \{y\}$ induces an odd antihole. So it must be that the number of T-edges in xPz and the length of xPz have the same parity. The same holds for zPy. So the number of T-edges in P and the length of P have the same parity, and we have outcome (i) or (ii).

Case 2: T induces a stable set. We denote by ε the parity of the length of P. Mark the vertices of P that have at least one neighbor in T. Call an *interval* any subpath of P, of length at least 1, whose ends are marked and whose internal vertices are not. Since x and y are marked, the edges of P are partitioned by the intervals of P.

We claim that every interval of P either has even length or has length 1. Indeed, suppose there is an interval of odd length, at least 3, say $P' = x'' \dots y''$, named so that x, x'', y'', y appear in this order along P. Let u and v be neighbors of x'' and y'' in T, respectively. If x'' and y'' have a common neighbor t in T then $P' \cup \{t\}$ induces an odd hole. Hence $u \neq v$, $x'' \neq x$, $y'' \neq y$, v is not adjacent to x'', and u is not adjacent to y''. If $x'' \neq x'$, then $P' \cup \{u, x, v\}$ induces an odd hole. So, $x'' = x'$ and similarly, $y'' = y'$. Hence, $\{u, v\}$ is a leap, a contradiction. This proves our claim.

Hence, the number of intervals of length 1 in P has parity ε. Moreover, we claim that for every interval of length 1, there is a vertex in T adjacent to both its ends. Indeed, suppose that there is an interval $x''y''$ such that x'' and y'' do not have a common neighbor in T. Let u be a neighbor of x'' in T, and let v be a neighbor of y'' with $u \neq v$, $uy'' \notin E(G)$, and $vx'' \notin E(G)$. Note that $x \neq x''$ and $y \neq y''$. If $x'' \neq x'$, then $\{u, x, v, x'', y''\}$ induces an odd hole. So, $x'' = x'$ and similarly $y'' = y'$. Now $\{u, v\}$ is a leap, a contradiction.

For every $v \in T$, denote by $f(v)$ the set of all $\{v\}$-complete edges of P. Let $v_1, \dots, v_n$ be the elements of T. We know that $|f(v_1) \cup \dots \cup f(v_n)|$ has parity ε, since, from the previous paragraph, it is equal to the number of the intervals of length 1. Moreover, by the sieve formula we have:

$$\begin{aligned} |f(v_1) \cup \dots \cup f(v_n)| &= \sum_i |f(v_i)| \\ &\quad - \sum_{i \neq j} |f(v_i) \cap f(v_j)| \\ &\quad \vdots \end{aligned}$$

$$
\begin{aligned}
&+(-1)^{(k+1)} \sum_{I \subset \{1,\dots,n\}, |I|=k} |\cap_{i\in I} f(v_i)| \\
&\vdots \\
&+(-1)^{(n+1)} |f(v_1) \cap \cdots \cap f(v_n)|
\end{aligned}
$$

By the induction hypothesis, we know that if $S \subsetneq T$, then the number of S-complete edges in P has parity ε. Hence if $I \subsetneq \{1, \dots, n\}$, then $|\cap_{i\in I} f(v_i)|$ has parity ε. Thus, we can rewrite the above equality modulo 2 as:

$$
|f(v_1) \cup \cdots \cup f(v_n)| = (2^n - 2) + (-1)^{(n+1)} |f(v_1) \cap \cdots \cap f(v_n)|
$$

Since $|f(v_1) \cup \cdots \cup f(v_n)|$ has parity ε, it follows that $|f(v_1) \cap \cdots \cap f(v_n)|$ has parity ε, meaning that the number of T-edges in P has parity ε. It follows that one of (i) of (ii) holds.

Case 3: We are neither in Case 1 nor in Case 2 (so T is not a stable set and there is no T-complete vertex in P^*). Let $Q = u \cdots v$ be a longest path of $\overline{G}[T]$. So Q has length at least 2 (since T is not a stable set), and $T \setminus \{u\}$ and $T \setminus \{v\}$ are anticonnected sets. By the induction hypothesis, we know that P has an odd number of $T \setminus \{u\}$-edges and an odd number of $T \setminus \{v\}$-edges. Note that a $T \setminus \{u\}$-edge and a $T \setminus \{v\}$-edge have no common vertex, for otherwise there would be a T-complete vertex in P^*. In particular all $T \setminus \{u\}$-edges and $T \setminus \{v\}$-edges are different.

Suppose that Q has even length. Let $x_u x'_u$ be a $T \setminus \{u\}$-edge of P and $y'_v y_v$ be a $T \setminus \{v\}$-edge of P such that, without loss of generality, $x, x_u, x'_u, y'_v, y_v, y$ appear in this order on P. If x'_u is non-adjacent to y'_v then $\{x'_u, y'_v\} \cup Q$ induces an odd antihole. If $x \neq x_u$ then $\{x_u, y'_v\} \cup Q$ induces an odd antihole. If $y_v \neq y$ then $\{x'_u, y_v\} \cup Q$ induces an odd antihole. It follows that $P = x_u x'_u y'_v y_v$, but then $P \cup Q$ induces an odd antihole. Thus Q has odd length (at least 3).

Suppose that $T \setminus \{u, v\}$ is not anticonnected. Since $T \setminus \{u\}$ and $T \setminus \{v\}$ are anticonnected, there exists a vertex w in an anticomponent of $G[T \setminus \{u, v\}]$ that does not contain Q^* and such that w is adjacent in $\overline{G}$ to at least one of u, v; but then $Q \cup \{w\}$ induces in $\overline{G}[T]$ either a chordless path longer than Q or an odd hole, a contradiction. So $T \setminus \{u, v\}$ is anticonnected.

Now we know that there is an odd number of $T \setminus \{u, v\}$-edges in P (by the induction hypothesis). Recall that P has an odd number of $T \setminus \{u\}$-edges, an odd number of $T \setminus \{v\}$-edges, and that these are different, so these account for an even number of $T \setminus \{u, v\}$-edges; thus P has at least one $T \setminus \{u, v\}$-edge $x''y''$ that is neither a $T \setminus \{u\}$-edge nor a $T \setminus \{v\}$-edge. We

may assume that x, x'', y'', y appear in this order along P and that $y'' \in P^*$. So y'' is non-adjacent to one of u, v, say v. Then y'' is adjacent to u, for otherwise $Q \cup \{y''\}$ would induce an odd antihole. Then x'' is non-adjacent to u, for otherwise $x''y''$ would be a $T \setminus \{v\}$-edge. Then x'' is adjacent to v, for otherwise $Q \cup \{x''\}$ would induce an odd antihole. Then $x'' = x'$ for otherwise $Q \cup \{x'', y'', x\}$ would induce an odd antihole, and similarly $y'' = y'$. So $P = xx''y''y$ and $Q \cup \{x'', y''\}$ is a chordless odd path of $\overline{G}$, and we have outcome (iv). □

It is not easy to see how useful Lemma 7.1 is, so let us give now a simple application, that is 3.1 from [20].

Lemma 7.2 *In a Berge graph, if a hole C and an antihole D both have length at least 8, then $|V(C) \cap V(D)| \leq 3$.*

Proof. It is easy to check that P_4 is the only graph H on at least four vertices such that both H and $\overline{H}$ are subgraphs of some path. It follows that if $|V(C) \cap V(D)| \geq 4$, then $V(C) \cap V(D)$ induces a P_4, say $abcd$. So, there is a path P from a to d in G, such that $abcdPa$ is the hole C; and there is an antipath Q from b to c such that $bQcadb$ is the antihole D. Note that P and Q are both of odd length, at least 5.

The ends of P are Q^*-complete. If P^* contains a Q^*-complete vertex v, then $\{v\} \cup V(Q)$ induces an odd antihole. Therefore, by Lemma 7.1 $Q*$ contains a leap $\{u, v\}$ for P, so some path P' from u to v has the same interior as P. Observe that u and v are consecutive along Q, and because of the length of Q, one of b, c (say b) is complete to $\{u, v\}$. It follows that $V(P') \cup \{b\}$ induces an odd hole. □

We now give a corrolary of 7.1 that is used constantly in [20] (where it is called 2.2).

Corollary 7.3 *Let T be an anticonnected set of vertices in a Berge graph G. If P is a path with odd length at least 3, vertex-disjoint from T, whose ends are T-complete and such that no internal vertex of P is T-complete, then every T-complete vertex has a neighbor in P^*.*

Proof. Let v be a T-complete vertex, and suppose that v has no neighbor in P^* (so v is not in $V(P) \cup T$). Apply Lemma 7.1 to T and P. If outcome (iii) of Lemma 7.1 holds, then a path of odd length with same interior as P joins the members of the leap. Together with v, it forms an odd hole, a contradiction. If outcome (iv) of Lemma 7.1 holds, then the antipath of odd length can be completed to an odd antihole through v, a contradiction. □

Further reading

Another proof of the Roussel–Rubio lemma is given in Maffray and Trotignon [73], where some applications are also presented. The proof given here in the case when T is a stable set is due to Kapoor, Vušković and Zambelli, see [93] where several very simple applications are presented.

8 Book from the Proof

The goal of this section is to present a self-contained lemma of [20] in order to give some taste of the technicalities. Let us replace this lemma in its context by stating informally how line graphs are handled.

A line graph K of a bipartite graph is formed of a bunch of cliques, and a bunch of paths linking them. If K is contained in a Berge graph G, a vertex $v \in V(G \setminus K)$ is *major* if it has many (here at least 2) neighbors in each of the cliques, and *minor* if it has neighbors in at most one of the paths, or in at most one of the cliques. An important result is that every vertex is major or minor, or allows one to obtain a larger line graph. This is not fully true (Fig. 4 contains counter-examples), but vertices that are neither major nor minor can be considered as part of some of the paths in what is called a generalized line graph. Therefore, when the generalized line graphs are properly defined, it is true that every vertex is major or minor w.r.t. a maximal generalized line graph. A next step is to prove that connected (sometimes anticonnected) components of vertices of $G \setminus K$ behave in fact as a vertex. A reader wanting to know more details for components can read the proof of 10.1 in [20] and we deal with anticomponents below.

If there are no major vertices, the graph is formed of the line graph, and possibly a bunch of minor components. If some of these components attach to a clique, then there is a balanced skew partition, and if some attach to a path, there is a 2-join. If there are major vertices, then it can be proved that an anticomponent of these is formed of vertices that all major *in the same way*, meaning that they all attach to exactly the same vertices of the cliques. Therefore, an anticomponent of major vertices is complete to its attachment, and forms a skew cutset separating parts of the line graph. The next lemma (it is a variant of 7.3 from [20]) proves a statement of this form. It illustrates another breakthrough made in [20]: how the Roussel-Rubio lemma can be used to find skew partitions.

Lemma 8.1 *In a Berge graph G, let K be a prism with triangles $a_1a_2a_3$ and $b_1b_2b_3$ and paths $P_i = a_i \dots b_i$, $i = 1, 2, 3$. Suppose that for $i = 1, 2, 3$, P_i*

has length at least 2. Let Y be an anticonnected set of vertices such that each of them have at least two neighbors in $\{a_1, a_2, a_3\}$ and at least two neighbors in $\{b_1, b_2, b_3\}$ (so Y is disjoint from K). Then at least two members of $\{a_1, a_2, a_3\}$ and at least two members of $\{b_1, b_2, b_3\}$ are Y-complete.

Proof. Suppose not; then there is an antipath with interior in Y joining two vertices both in $\{a_1, a_2, a_3\}$ or both in $\{b_1, b_2, b_3\}$. Let Q be a shortest such antipath. Suppose up to symmetry that Q is from a_1 to a_2. Every vertex in Y is adjacent to either a_1 or a_2, so Q has length at least 3. From the minimality of Q, a_3 is Q^*-complete, and so is at least one of $\{b_1, b_2, b_3\}$, say b_i. Since Q can be completed to an antihole via $a_1 b_i a_2$, it follows that Q has even length, therefore at least 4. We set $Q = a_1 q_1 \dots q_n a_2$. Let a'_i be the neighbor of a_i in P_i, $i = 1, 2$ and $P = a'_1 P_1 b_1 b_2 P_2 a'_2$.

(1) At least one internal vertex of P is $(Q^* \setminus q_1)$-complete and at least one internal vertex of P is $(Q^* \setminus q_n)$-complete.

If one of b_1 or b_2 is Q^*-complete, the claim is obviously true. Otherwise, none of b_1, b_2 is Q^*-complete so there exists a antipath from b_1 to b_2 whose interior in Q^*, and from the minimality of Q this antipath has the same interior as Q. It follows that one of b_1, b_2 is complete to $Q^* \setminus q_1$ and the other one is complete to $Q^* \setminus q_n$. This proves (1).

(2) If an internal vertex of P is $(Q^* \setminus q_1)$-complete or $(Q^* \setminus q_n)$-complete, then it is Q^*-complete. If a'_1 is $(Q^* \setminus q_1)$-complete, then it is Q^*-complete.

If an internal vertex v of P is $(Q^* \setminus q_n)$-complete, then it is Q^*-complete for otherwise, $v q_n Q a_1 v$ is an odd antihole. If v is a'_1 or is an internal vertex of P, and v is $(Q^* \setminus q_1)$-complete, then v is Q^*-complete, for otherwise, $v q_1 Q a_2 v$ is an odd antihole. This proves (2).

If both a'_1, a'_2 are Q^*-complete, then Q can be completed to an odd antihole via $a_1 a'_2 a'_1 a_2$, a contradiction. So, up to symmetry, we suppose from here on that a'_1 is not Q^*-complete. It follows by (2) that a'_1 is not $Q^* \setminus q_1$ complete.

Call x the $Q \setminus q_1$ complete vertex of P closest to a'_1 along P. By (1), x exists and is an internal vertex of P. By (2), x is in fact Q^*-complete.

If $a'_1 a_1 P x$ has odd length, then by Corollary 7.3 applied to $a_1 a'_1 P x$ and $Q^* \setminus q_1$, there is a contradiction because a_3 is $(Q^* \setminus q_1)$-complete and has no neighbor in the interior $a_1 P x$. So $a_1 P x$ has even length. It follows that $x P a'_2 a_2$ has odd length. By Corollary 7.3 applied to $x P a'_2 a_2$ and $Q^* \setminus q_n$, and because of a_3, there must be an internal vertex v of $x P a'_2 a_2$ that is $(Q^* \setminus q_n)$-complete. By (2), v is in fact Q^*-complete. But then, $a_3 a_1 a'_1 P x$ is

odd, its ends are Q^*-complete, but none of its internal vertex is Q^*-complete. This contradicts Corollary 7.3 because of v. $\square$

The lemma above shows how the Roussel–Rubio lemma 'generates' antiholes to provide contradictions. Maybe it can be used to invistigate the structure of several subclasses of Berge graphs, for instance the next one.

Question 8.2 *Is there a structural characterization of Berge graphs with no antihole of length at least 6?*

Further reading

About technicalities, the best reading is of course Chudnovsky, Robertson, Seymour and Thomas [20]. The paper is very well organized: the first four sections are devoted to technical lemmas that are used extensively in the sequel. But as a result of this wise organization, it is very hard to extract a meaningful part: it is difficult to feel how useful the technical lemmas are without knowing the sequel, and the sequel is difficult to understand without mastering the technicalities of the lemmas. A reader seeking the easiest parts from [20] should try 15.1, Section 15 or Section 16. Section 10 is devoted to the even prism and is a self-contained chunk that has fewer technicalities than the rest of the paper (but still relies a lot on technical lemmas of the previous sections).

A reader might want to read self-contained papers using the same kind of technicalities as [20] in simpler situations. For attachement to line graphs, a possible reading is Lévêque, Maffray and Trotignon [65] (where Question 5.1 is partially answered by the way). About how attachments to a wheel can be used to decompose a class, Radovanović and Vušković [80] is a good reading.

9 Recognition of perfect graphs

From here on, we investigate works on perfect graphs that were done after the proof of the SPGT. Soon after the proof of the SPGT, another open question was solved. Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [17] found a polynomial time algorithm that decides whether an input graph is Berge. Observe that this algorithm is independent from the SPGT. It takes any graph G as an input and outputs in time $O(n^9)$ an odd hole-or-antihole of G (if any). We give here a brief outline (that is a copy from the introduction of [17]). In what follows, G is the input graph of the algorithm.

> [...] we would like to decide either that G is not Berge, or that G contains no odd hole. (To test Bergeness, we just run this algorithm on G and then again on the complement of G.) If there is an odd hole in G, then there is a shortest one, say C. A vertex of the remainder of G is C-major if its set of neighbours in C is not a subset of the vertex set of any 3-vertex path of C; and C is *clean* (in G) if there are no C-major vertices in G. If there happens to be a clean shortest odd hole in G, then it stands out and can be detected relatively easily; and that essentially is the first step of our algorithm, a routine to test whether there is a clean shortest odd hole. The remainder of the algorithm consists of reducing the general problem to the 'clean' case that was just handled. If C is a shortest odd hole in G, let us say a subset X of $V(G)$ is a *cleaner* for C if $X \cap V(C) = \emptyset$ and every C-major vertex belongs to X. Thus if X is a cleaner for C then C is a clean hole in $G \setminus X$. The idea of the remainder of the algorithm is to generate polynomially many subsets of $V(G)$, such that if there is a shortest odd hole C in G, then one of the subsets will be a cleaner for C. If we can do that, then we delete each of these subsets in turn, thereby generating polynomially many induced subgraphs; and we know that there is an odd hole in G if and only if in one of these subgraphs there is a clean shortest odd hole. Thus we can decide whether G has an odd hole by testing whether any of these subgraphs has a clean shortest odd hole.

Theorem 9.1 (Chudnovsky, Cornuéjols, Liu, Seymour and Vušković 2002) *There exists an algorithm that decides whether an input graph is Berge in time $O(n^9)$.*

Let us now explain briefly the two steps from the sketch above. The second step relies on a powerful technique discovered by Conforti and Rao [35], called *cleaning*, which consists in 'guessing' a cleaner. Here, the way to guess the cleaner comes from Roussel–Rubio like lemmas, saying that the set X of vertices vertices to be 'guessed', that are all major vertices of some possible smallest odd hole, are all common neighbors of some set S (of size bounded by constant) of neighbors. Therefore, one can enumerate all possible S's by brute-force, and for each of them nominate X as the set of S-complete vertices.

The first step relies on the *shortest path detector*, a method designed by Chudnovsky and Seymour [17] that is used twice in the algorithm: once to

detect a clean odd hole, and once to detect a pyramid. Detecting a pyramid is needed in a preprocessing step that we omitted to explain in the sketch above. It consists in the detection of several substructures certifying that the graph is not Berge, and one of them is the pyramid (that entails an odd hole).

To explain the shortest path detector, we give here a simpler algorithm that detects prisms in graphs with no pyramids (recall that in general graphs, the problem is NP-complete). This algorithm is not published because a faster one exists, see [72]. A shortest path detector always relies on a lemma stating in the smallest substructure of the kind that we are looking for, a path linking two particular vertices of the substructure can be replaced by any shortest path.

Lemma 9.2 *Let G be a graph with no pyramid. Let K be a smallest prism in G. Suppose that K is formed by paths P_1, P_2, P_3, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, so that, for $i = 1, 2, 3$, path P_i is from a_i to b_i. Then:*

If R_i is any shortest path from a_i to b_i whose interior vertices are not adjacent to a_{i+1}, a_{i+2}, b_{i+1} or b_{i+2}, then R_i, P_{i+1}, P_{i+2} form a prism on $|V(K)|$ vertices in G, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ (the addition of subscripts is taken modulo 3).

Proof. Suppose that the lemma fails for say $i = 1$. So, some interior vertex of R has neighbors in the interior of P_2 or P_3. Let x be such a vertex, closest to a_1 along R. Let a'_2 (resp. a'_3) be the neighbor of a_2 (resp. a_3) along P_2 (resp. P_3). Let $Q = a'_2 P_2 b_2 b_3 P_3 a'_3$. Let y (resp. z) be the neighbor of x closest to a'_2 (resp. a'_3) along Q.

If $y = z$ then $yxRa_1$, yQa'_2a_2 and yQa'_3a_3 form a pyramid, a contradiction. If $y \neq z$ and $yz \notin E(G)$ then xRa_1, $xyQa'_2a_2$ and $xzQa'_3a_3$ form a pyramid, a contradiction. If $yz \in E(G)$ then xRa_1, yQa'_2a_2 and zQa'_3a_3 form a prism on less vertices than K, a contradiction. □

Now detecting a prism in a graph with no pyramid can be performed as follows. For all 6-tuples $(a_1, a_2, a_3, b_1, b_2, b_3)$ compute three shortest paths R_i in $G \setminus ((N[a_{i+1}] \cup N[a_{i+2}] \cup N[b_{i+1}] \cup N[b_{i+2}]) \setminus \{a_i, b_i\})$ from a_i to b_i. Check whether R_1, R_2, R_3 form a prism, and if so output it. If no triple of paths forms a prism, output that the graph contains no prism. If the algorithm outputs a prism, this is obviously a correct answer: the graph contains a prism. Suppose conversely that the graph contains a prism. Then it contains a smallest prism with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. At some step, the algorithm will check this 6-tuple (unless a prism is discovered before, but

then the correctness is proved anyway). By three applications of Lemma 9.2, we see that the three paths R_1, R_2, R_3 form a prism that is output. All this take time $O(n^8)$.

The following is still open.

Question 9.3 *Is there a polynomial time algorithm to decide whether an input graph has an odd hole?*

Further reading

In [17], another algorithm for recognizing Berge graphs is given. It relies on decompositions and uses Theorem 4.2. Detecting odd holes can be solved in polynomial time under the assumption that the largest size of a clique in the input graph is bounded by some constant. This is proved by Conforti, Cornuéjols, Liu, Vušković and Zambelli [32] that is a good reading for understanding the main ideas of the recognition of Berge graphs. A combination of cleaning and shortest path detector is used by Chudnovsky, Seymour and Trotignon [26] to decide in polynomial time whether a graph contains a subdivision of the net (the *net* is the graph obtained from a triangle by adding a pendant edge at each vertex).

10 Berge trigraphs

An obvious question about Theorem 6.1 is whether it is best possible. Is each outcome necessary? Are there outcomes that can be made more precise? A careful reader of [20] might notice that the outcome 'homogeneous pair' is obtained only once in the whole proof, and might therefore wonder whether it is really necessary. Chudnovsky proved it is not. Let us explain how.

A way to proceed is to consider a smallest Berge graph G such that the homogeneous pair is the only outcome satisfied by G in Theorem 6.1, and to look for a contradiction. A natural idea is then to 'contract' a homogeneous pair (A, B) in order to find a smaller Berge graph G'. Then, apply Theorem 6.1 to G', and prove that any outcome of Theorem 6.1 in G' yields an outcome in G (because G' and G are very similar). From the initial assumption, G' therefore satisfies no outcome of Theorem 6.1, and this provides a contradiction. We call this method *bootstrap*, because it improves structural theorems somehow for free. The natural way to 'contract' a homogeneous pair (A, B) of G is to replace A (resp. B) by a vertex a (resp. b) complete to $C(A)$ (resp. $C(B)$) and anticomplete to $\overline{C}(A)$ (resp. $\overline{C}(B)$). The bootstrap method is hard to implement in this context. The problem is with

ab: should it be an edge or an antiedge of G'? Both choices lead to difficult technicalities: if ab is chosen to be an antiedge, it could be that that some skew cutset separates a from b in G', while A and B are linked in G. If a is chosen to be adjacent to b, it could be that the same phenomenon happens in the complement. No neat example of these bad phenomena can be given, because as we will see, it is true that the outcome 'homogeneous pair' is not necessary in Theorem 6.1. But any attempt of proof will face this issue and is likely to fail.

The idea of Chudnovsky is to leave undecided the adjacency between a and b in G'. To this purpose, she defines *trigraphs* as graphs with edges, antiedges, and a third kind of adjacency: *switchable pair*. For every pair of distinct vertices x and y, xy is an edge, an antiedge or a switchable pair. A *realization* G of a trigraph T is any graph on $V(T)$ such that all edges (resp. antiedges) of T are edges (resp. antiedges) of G (so, every switchable pair is transformed into an edge or an antiedge). A trigraph is *Berge* if every realization is Berge. We do not give the long list of definitions that translates naturally the vocabulary of graphs to trigraphs. The key point is in the definitions of the decompositions: all the 'important' edges in definitions of decompositions are not allowed to be switchable pairs. For instance, if (X, Y) is a 2-join of a trigraph T, no switchable pair of T is from X to Y. If (X, Y) is a skew partition of a trigraph T, it must be that X is partitioned into X_1 and X_2 such that no edge and no switchable pair exists between X_1 and X_2, and it must be that Y is patitioned into Y_1 and Y_2 such that no antiedge and no switchable pair exists between Y_1 and Y_2. It is easy to guess how useful is this requirement: for instance, when building G' from G as in the paragraph above, ab is defined to be a switchable pair. So, the problem that we mentioned with the skew cutset separating a from b does not exists anymore.

Of course, the slight problem with this notion of trigraph is that the proof of Theorem 6.1 has to be made again from the beginning for Berge trigraphs. Chudnovsky proved several decomposition theorems similar to Theorem 6.1 for Berge trigraphs. The proof of the main one is self-contained and runs along more than 200 pages. We do not give the precise statements here, it would be pointless since we do not give the precise definitions of decompositions and basic classes of trigraph. The precise statements of the theorems are in [15], where only parts of the proofs are given. The complete proof is in [14]. With trigraphs, the bootstrap method works smoothly and yields the following (that we translate back to graphs, since a graph is a particular trigraph).

Theorem 10.1 (Chudnovsky 2003) *Every Berge graph is basic, or has a 2-join, a complement 2-join or a balanced skew partition.*

An interesting feature of trigraphs is that additional conditions can be added. For instance, a *monogamous* trigraph is a trigraph such that every vertex is member of at most one switchable pair. Monogamous trigraphs are very convenient to handle the interactions of odd 2-joins and homogenous pairs. When a monogamous trigraph T has a homogeneous pair (A, B), a smaller trigraph T' can be constructed as follows: replace A (resp. B) by a vertex a (resp. b) strongly complete to $C(A)$ (resp. $C(B)$) and strongly anticomplete to $\overline{C}(A)$ (resp. $\overline{C}(B)$) (here *strongly* means that only real edges are used, not switchable pairs), and link a to b by a switchable pair. When a monogamous trigraph T has an odd 2-join (X_1, X_2) with sets $A_1, B_1, C_1, A_2, B_2, C_2$ as in the definition, a smaller trigraph T_2 can be constructed as follows: delete C_1 replace A_1 (resp. B_1) by a vertex a_1 (resp. b_1) strongly complete to A_2 (resp. B_2) and strongly anticomplete to $B_2 \cup C_2$ (resp. $A_2 \cup C_2$), and link a_1 to b_1 by a switchable pair. When applied iteratively, this way of constructing blocks of decompositions preserves the property of being monogamous, and shows that all the decompositions used in the process do not cross. A slightly more general notion of trigraph is defined in [28] to handle interactions between 2-joins, complement 2-joins and homogeneous pairs.

Another potential improvement of Theorem 6.1 is about the technical requirements in the definition of a 2-join. Some authors consider only *non-path 2-joins* (already defined page 10). In some applications, it is essential to use non-path 2-joins, because one needs to replace one side of the 2-join by a long path to recurse (in proof by induction, or in algorithms), and this obviously fails if the side is already a long path. Trotignon [92] investigated this question and obtained the following result (with the bootstrap method for graphs starting from Theorem 10.1). *Path cobipartite* graphs and *path double-split graphs* are Berge graphs obtained by subdividing edges in complement of bipartite graphs and in double split graphs respectively (a more precise definition can be given, but this one is enough here). Observe that subdividing an edge in any graph creates a path 2-join.

Theorem 10.2 (Trotignon 2008) *If G is a Berge graph, then G is basic, or one of G, $\overline{G}$ is a path-cobipartite graph, or one of G, $\overline{G}$ is a path-double split graph, or one of G, $\overline{G}$ has a non-path 2-join, or G has a balanced skew partition, or G has a homogeneous pair and a path 2-join.*

Observe that this theorem shows that homogeneous pairs are not necessary to decompose Berge, that path 2-joins are not necessary to decompose Berge graphs, but does not show that one can get rid of both outcomes. We now give examples showing that every outcome of Theorem 10.2 is needed. In Fig. 6 and 7, a path cobipartite graph and a path double-split graph are represented. They both have a path 2-join, showing that to get rid of path 2-joins, the two new basic classes are really needed. The graph represented in Fig. 8 is interesting because it is decomposable only by a path 2-join or by a homogeneous pair, showing that it is impossible to get rid of both outcomes. Obtaining graphs uniquely decomposable by a 2-join is easy by the following recipe. Consider a Berge graph G_i ($i = 1, 2$) that contains a path P_i of length 3 from a_i to b_i and such that $A_i = N(a_i)$ and $B_i = N(b_i)$ are disjoint. Now take the disjoint union of G_1 and G_2, and add all possible edges between A_1 and A_2 and between B_1 and B_2. The resulting graph obviously has a 2-join, and if G_1 and G_2 are sufficiently general, this is the only decomposition. This recipe can be applied to the graphs from Fig. 6 and 7 for instance. On Fig. 10, a Berge graph uniquely decomposable by a balanced skew partition is represented.

Not much work has been devoted to the algorithmic aspects of trigraphs. The following is still open.

Question 10.3 *What is the complexity of recognizing Berge trigraphs?*

Further reading

It seems now that trigraphs are an important general tool in structural graph theory, as suggested by their use in the study of claw-free graphs (see Chudnovsky and Seymour [22]) and bull-free graphs (see Chudnovsky [16]). About algorithms for Berge trigraphs, Chudnovsky, Trotignon, Trunck and Vušković [28] seems to be the only available reference.

11 Even pairs: a shorter proof of the SPGT

An *even pair* in a graph is a pair $\{x, y\}$ of vertices such that every path between them has even length. This notion is involved in algorithmic aspects of perfect graphs and allows to significantly shorten the proof of the SPGT. Meyniel [75] proved the following.

Theorem 11.1 (Meyniel 1987) *A minimally imperfect graph has no even pair.*

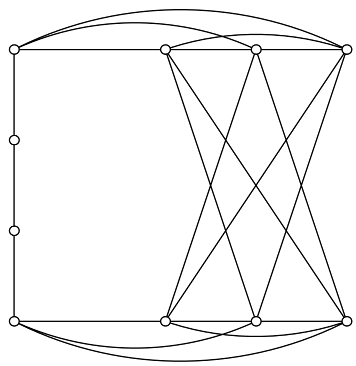

Figure 6: A path-cobipartite graph

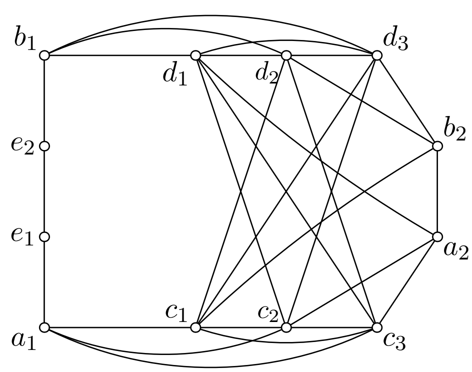

Figure 7: A path-double split graph

Therefore, even pairs could be an ingredient of a useful decomposition theorem for Berge graphs. Given two vertices x, y in a graph G, the operation of *contracting* them means removing x and y and adding one vertex with edges to every vertex of $G \setminus \{x, y\}$ that is adjacent in G to at least one of x, y; we denote by G/xy the graph that results from this operation. Fonlupt and Uhry [45] proved that *if G is a perfect graph and $\{x, y\}$ is an even pair in G, then the graph G/xy is perfect and has the same chromatic number as G.* In particular, given a $\chi(G/xy)$-colouring c of the vertices of G/xy, one can easily obtain a $\chi(G)$-colouring of the vertices of G as follows: keep the colour for every vertex different from x, y; assign to x and y the colour assigned by c to the contracted vertex. This idea could be the basis for a conceptually simple colouring algorithm for Berge graphs: as long as the graph has an even pair, contract any such pair; when there is no even pair find a colouring c of the contracted graph and, applying the procedure above repeatedly, derive from c a colouring of the original graph. The algorithm for recognizing Berge graphs mentioned at the end of the preceding paragraph can be used to detect an even pair in a Berge graph G; indeed, it is easy to see that two non-adjacent vertices a, b form an even pair in G if and only if the graph obtained by adding a vertex adjacent only to a and b is Berge. Thus, given a Berge graph G, one can try to colour its vertices by keeping contracting even pairs until none can be found. Then some questions arise: what are the Berge graphs with no even pair? What are, on the contrary, the graphs for which a sequence of even-pair contractions leads to graphs that are trivially easy to colour?

Bertschi [8] proposed the following definitions. A graph G is *even contractile* if either G is a clique or there exists a sequence $G_0, \ldots, G_k$ of graphs such that $G = G_0$, for $i = 0, \ldots, k-1$ the graph G_i has an even pair $\{x_i, y_i\}$ such that $G_{i+1} = G_i/x_iy_i$, and G_k is a clique. A graph G is *perfectly contractile* if every induced subgraph of G is even contractile.

A simple observation is that odd holes and antiholes of length at least 5 have no even pairs. Also some Berge graphs have no even pairs, such as $\overline{C_6}$, $L(K_{3,3}) \setminus e$, and more generally sufficiently connected line graphs of bipartite graphs (see Hougardy [56] for more about that), and every even antihole of length at least 6. In fact every known example of a Berge graph with no even pair either contains an odd prism or an antihole. Odd prisms different from $\overline{C_6}$ have even pair, but they are not perfectly contractile (any attempt to contract them leads to $\overline{C_6}$). This justifies the following definitions. *Grenoble graphs* are these graphs with no odd holes, no antihole of length at least 5 and no odd prism. *Artemis graphs* are these graphs with no odd holes, no antiholes of length at least 5 and no prisms.

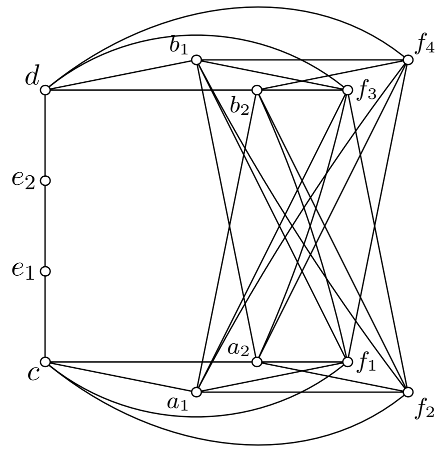

Figure 8: A graph that has a homogeneous pair $(\{a_1, a_2\}, \{b_1, b_2\})$ and a path 2-join

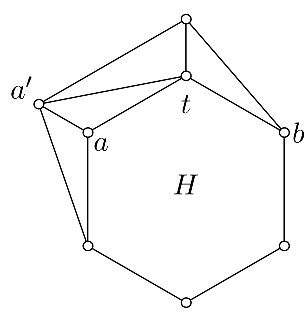

Figure 9: An artemis graph

To prove the existence an even pair in an Artemis graph, an idea is to consider a shortest hole H, and two vertices u, v at distance 2 along H, with a common neighbor $t \in V(H)$. As shown in Fig. 9, $\{a, b\}$ may fail to be an even pair because of a' which is the second vertex of a P_4 from a to b. However, one may complain that H is not the 'best' hole, the hole H' obtained by replacing a by a' is better, because $\{a', b\}$ is an even pair of G. In this sense, a' is 'better' than a. In the square-free case, Linhares and Maffray [66] showed that the neighborhood of t contains two cliques A and B. On A (and also on B), there exists an order (corresponding to the idea of a vertex better than another one). A maximal element in A and a maximal element in B forms an even pair of G. This method was extended by Maffray and Trotignon [73] to general Artemis graphs by using the Roussel–Rubio lemma.

Theorem 11.2 (Maffray and Trotignon 2005) *Every Artemis graph is perfectly contractile.*

This theorem can be transformed into a colouring algorithm [64] that generalizes several well known algorithms for colouring classes of perfect graphs, such as Meyniel graphs and weakly chordal graphs. It yields a combinatorial polynomial time colouring algorithm for perfectly orderable graphs (none was known before).

Theorem 11.3 (Lévêque, Maffray, Reed and Trotignon 2009) *There exits an algorithm that colours every Artemis graph in time $O(n^2m)$.*

The proof of the next theorem from [24] uses some technics of the proof of Theorem 11.3, is very readable, and as we will see saves about 50 pages in the proof of the SPGT. An *odd wheel* (C, T) in a graph G consists of a hole C of length at least six, and a nonempty anticonnected subset $T \subseteq V(G) \setminus V(C)$, such that at least three vertices of C are T-complete, and there is a path P of C with odd length at least 3, such that its ends are not T-complete and all its internal vertices are T-complete. A *long prism* is a prism such that at least one the paths has length at least 2. Let us say that G is impoverished if G is Berge, and G and $\overline{G}$ both contain no odd wheel, long prism or double diamond. A *dominant pair* in G is a pair (x, y) of nonadjacent vertices such that every other vertex of G is adjacent to at least one of x, y.

Theorem 11.4 (Chudnovsky and Seymour, 2009) *If G is impoverished, then either G admits a star cutset or an even pair or a dominant pair, or G is a complete graph.*

It is not very difficult to prove that a minimally imperfect graph has no dominant pair (see [24]), so it follows easily by Theorem 11.1 and Lemma 4.1 that every impoverished graph is perfect. Since the last 55 pages of the proof of the SPGT are devoted to finding a skew partition in an impoverished graph, they are no longer necessary and can be replaced by the 8 pages needed to prove Theorem 11.4.

A neat generalization of Theorem 11.3 is the following conjecture (see [43]).

Conjecture 11.5 (Everett and Reed) *Every Grenoble graph is perfectly contractile.*

To prove perfection for a class $\mathcal{C}$, a statement of the following form is enough: every graph in $\mathcal{C}$ has an even pair, or its complement has an even pair. Note that this does not hold for all Berge graphs as shown by $L(K_{3,3} \setminus e)$ (see Fig. 5). It seems that the only known theorem of this form is the following from [39]. The *bull* is the graph obtained from the triangle by adding two pending edges at different vertices.

Theorem 11.6 (de Figueiredo, Maffray and Porto, 1997) *If G is a bull-free Berge graph with at least two vertices then at least one of G or $\overline{G}$ has an even pair.*

A graph G is *bipartisan* if it is Berge and contains no $L(K_{3,3} \setminus e)$, no double diamond, and none of $G, \overline{G}$ contains a long prism (a prism is *long* when at least one of three paths of the prism has length at least 2). Observe that every bull-free Berge graph is bipartisan. An intermediate result of [20] is that a bipartisan graph is bipartite, complement of bipartite, or has a balanced skew partition. A direct proof of the following conjecture (published in [12] and [24]) would generalize Theorem 11.6 and could shorten the proof of the SPGT.

Conjecture 11.7 (Maffray, 2002) *If G is bipartisan with at least two vertices, then one of G, $\overline{G}$ contains an even pair.*

Further reading

A good survey on even pairs is Everett, de Figueiredo, Linhares Sales, Maffray, Porto and Reed [43]. It seems that even pairs is a conjecture-cornucopia: other conjectures on even pairs can be found in Burlet, Maffray and Trotignon [12] or in Lévêque and de Werra [62]. Very short proofs of the existence of even pairs in classical classes of perfect graphs (Meyniel graphs and weakly chordal graphs) can be found in Trotignon and Vušković [93].

12 Colouring perfect graphs

In the 1980s, Gröstchel, Lovász and Schrijver [51] devised a polynomial time algorithm that colours any input perfect graph. This algorithm relies on the ellipsoid method, and one may wonder whether a more combinatorial algorithm exists. Even if there is no formal definition of what a *combinatorial* algorithm should be, most mathematicians agree that an algorithm that relies on graphs searches and decompositions, and even on classical linear programming, can be called 'combinatorial' and that the ellipsoid method cannot. So this question is considered open. A more precise question is whether a fast colouring algorithm can be derived from Theorem 6.1. We investigate here recent progress in this direction.

We start with a combinatorial polynomial time colouring algorithm for perfect graphs, due to Grötschel, Lovász and Schrijver [51], under the assumption that a subroutine for computing a maximum weighted stable set is available. Here weights are not essential, because they can be simulated by replications, but we use them for convenience and because some subclasses of perfect graphs are not closed under replication. We suppose that $\mathcal{C}$ is a subclass of perfect graphs, and that there is an $O(n^k)$ algorithm $\mathcal{A}$ that computes a maximum weighted stable set and a maximum weighted clique for any input graph in $\mathcal{C}$ (so $\mathcal{C}$ needs not be closed under taking complement). Observe that we do not assume that $\mathcal{C}$ is closed under replication or even under taking induced subgraphs. In what follows, n denotes the number of vertices of the graph under consideration.

Lemma 12.1 *There is an algorithm with the following specification:*

Input: *A graph G in $\mathcal{C}$, and a sequence $K_1, \dots, K_t$ of maximum cliques of G where $t \leq n$.*

Output: *A stable set of G that intersects each K_i, $i = 1, \dots, t$.*

Running time: $O(n^k)$

Proof. By $\omega(G)$ we mean here the maximum *cardinality* of a clique in G. Give to each vertex v the weight $y_v = |\{i; v \in K_i\}|$. Note that this weight is possibly zero. With Algorithm $\mathcal{A}$, compute a maximum weighted stable set S of G.

Let us consider the graph G' obtained from G by replicating y_v times each vertex v. So each vertex v in G becomes a stable set Y_v of size y_v in G' and between two such stable sets Y_u, Y_v there are all possible edges if

$uv \in E(G)$ and no edges otherwise. Note that vertices of weight zero in G are not in $V(G')$. Note also that G' may fail to be in $\mathcal{C}$, but it is easily seen to be perfect. By replicating y_v times each vertex v of S, we obtain a stable set S' of G' of maximum cardinality.

By construction, $V(G')$ can be partitioned into t cliques of size $\omega(G)$ that form an optimal colouring of $\overline{G'}$ because $\omega(G') = \omega(G)$. Since by Theorem 2.3 $\overline{G'}$ is perfect, $|S'| = t$. So, in G, S intersects every K_i, $i \in \{1, \dots, t\}$. $\square$

Theorem 12.2 (Gröstchel, Lovász and Schrijver 1988) *There exists an algorithm of complexity $O(n^{k+2})$ whose input is a graph from $\mathcal{C}$ and whose output is an optimal colouring of G.*

Proof. As in the proof of Lemma 2.1, we only need to show how to find a stable set S intersecting all maximum cliques of G, since we can apply recursion to $G \setminus S$ (by giving weight 0 to vertices of S). Start with $t = 0$. At each iteration, we have a list of t maximum cliques $K_1, \dots, K_t$ and we compute by the algorithm in Lemma 12.1 a stable set S that intersects every K_i, $i \in \{1, \dots, t\}$. If $\omega(G \setminus S) < \omega(G)$ then S intersects every maximum clique, otherwise we can compute a maximum clique K_{t+1} of $G \setminus S$ (by giving weight 0 to vertices of S). This will eventually find the desired stable set, the only problem being the number of iterations. We show that this number is bounded by n.

Let M_t be the incidence matrix of the cliques $K_1, \dots, K_t$. So the columns of M_t correspond to the vertices of G and each row is a clique (we see K_i as row vector). We prove by induction that the rows of M_t are independent. So, we assume that the rows of M_t are independent and prove that this holds again for M_{t+1}.

The incidence vector x of S is a solution to $M_t x = \mathbf{1}$ but not to $M_{t+1} x = \mathbf{1}$. If the rows of M_{t+1} are not independent, we have $K_{t+1} = \lambda_1 K_1 + \cdots + \lambda_t K_t$. Multiplying by x, we obtain $K_{t+1} x = \lambda_1 + \cdots + \lambda_t \neq 1$. Multiplying by $\mathbf{1}$, we obtain $\omega = K_{t+1}\mathbf{1} = \lambda_1 \omega + \cdots + \lambda_t \omega$, so $\lambda_1 + \cdots + \lambda_t = 1$, a contradiction.

So the matrices $M_1, M_2, \dots$ cannot have more than n rows. Hence, there are at most $|V(G)|$ iterations. $\square$

By Theorem 12.2, we know that finding a maximum weighted stable set is enough to solve the colouring problem. Our question now is whether Theorem 6.1 (or one of its variants from Section 10) helps to find a maximum weighted stable set. It turns out that the question is easy for all basic classes

(for the historical ones, see [88] and for doubled graphs, see [28]). Also homogeneous pairs and complement 2-joins are not very hard to handle (see [28]).

For 2-joins, the situation is more complicated because a maximum stable set may overlap a 2-join in many ways. To illustrate the problem, we start with an NP-hardness result. We define a class $\mathcal{C}'$ of graphs for which computing a maximum stable set is NP-hard. The interesting feature of $\mathcal{C}'$ is that all graphs in $\mathcal{C}'$ are decomposable along 2-joins into one bipartite graph and several gem-wheels where a *gem-wheel* is any graph made of an induced cycle of length at least 5 together with a vertex adjacent to exactly four consecutive vertices of the cycle. Note that a gem-wheel is a line graph (of a cycle with one chord) and that computing a maximum weighted stable set in line graph $G = L(R)$ means computing a maximum weighted matching in R, which can be done by Edmonds's algorithm [42]. Therefore, the NP-completeness result below shows that being able to decompose along 2-joins into 'easy' graphs is not enough in general to compute stables sets.

A *flat* path in a graph is path whose internal vertices have degree 2 (in the graph). *Extending* a flat path $P = p_1 \dots p_k$ of a graph means deleting the interior vertices of P and adding three vertices x, y, z and the following edges: p_1x, xy, yp_k, zp_1, zx, zy, zp_k. By extending a graph G we mean extending all paths of $\mathcal{M}$ where $\mathcal{M}$ is a set a flat paths of length at least 3 of G. Class $\mathcal{C}'$ is the class of all graphs obtained by extending 2-connected bipartite graphs. From the definition, it is clear that all graphs of $\mathcal{C}'$ are decomposable along non-path 2-joins. One leaf of a decomposition tree will be the underlying bipartite graph. All the others leaves will be gem-wheels.

We call *4-subdivision* any graph G obtained from a graph H by subdividing four times every edge. More precisely, every edge uv of H is replaced by an induced path $uabcdv$ where a, b, c, d are of degree two. It is easy to see that $\alpha(G) = \alpha(H) + 2|E(H)|$. This construction, essentially due to Poljak [78], yields the next result observed by Naves (see [94]):

Theorem 12.3 (Naves, Trotignon and Vušković 2012) *The problem whose instance is a graph G from $\mathcal{C}'$ and an integer k, and whose question is 'Does G contain a stable set of size at least k' is NP-complete.*

Proof. Let H be any graph. First we subdivide 5 times every edge of H. So each edge ab is replaced by $P_7 = ap_1 \dots p_5b$. The graph H' obtained is bipartite. Now we build an extension G of H' by replacing all the P_5's $p_1 \dots p_5$ arising from the subdivisions in the previous step by P_4's. And for each P_4 we add a new vertex complete to it and we call *apex vertices* all

these new vertices. The graph G that we obtain is in $\mathcal{C}$. It is easy to see that there exists a maximum stable set of G that contain no apex vertex because an apex vertex of a maximum stable set can be replaced by one vertex of its neighborhood. So, we call G' the graph obtained from G by deleting all the apex vertices and see that $\alpha(G') = \alpha(G)$. Also, G' is the 4-subdivision arising from H. So from the remark above, maximum stable sets in H and G have sizes that differ by $2|E(H)|$. $\square$

We now explain how 2-joins can in fact help to find stable sets *in Berge graphs*. If (X, Y) is a 2-join of a graph G then let $X_1 = X$, $X_2 = Y$ and let A_1, B_1, C_1, A_2, B_2, C_2 be as in the definition of a 2-join. We define $\alpha_{AC} = \alpha(G[A_1 \cup C_1])$, $\alpha_{BC} = \alpha(G[B_1 \cup C_1])$, $\alpha_C = \alpha(G[C_1])$ and $\alpha_X = \alpha(G[X_1])$. Let w be the weight function on $V(G)$. When H is an induced subgraph of G, or a subset of $V(G)$, $w(H)$ denotes the sum of the weights of vertices in H. The following simple lemma describes the situation.

Lemma 12.4 *Let S be a maximum weighted strong stable set of G. Then exactly one of the following holds:*

(i) $S \cap A_1 \neq \emptyset$, $S \cap B_1 = \emptyset$, $S \cap X_1$ is a maximum weighted strong stable set of $G[A_1 \cup C_1]$ and $w(S \cap X_1) = \alpha_{AC}$;

(ii) $S \cap A_1 = \emptyset$, $S \cap B_1 \neq \emptyset$, $S \cap X_1$ is a maximum weighted strong stable set of $G[B_1 \cup C_1]$ and $w(S \cap X_1) = \alpha_{BC}$;

(iii) $S \cap A_1 = \emptyset$, $S \cap B_1 = \emptyset$, $S \cap X_1$ is a maximum weighted strong stable set of $G[C_1]$ and $w(S \cap X_1) = \alpha_C$;

(iv) $S \cap A_1 \neq \emptyset$, $S \cap B_1 \neq \emptyset$, $S \cap X_1$ is a maximum weighted strong stable set of $G[X_1]$ and $w(S \cap X_1) = \alpha_X$.

Proof. Follows directly from the definition of a 2-join. $\square$

The next inequalities are from [94]. They say how stable sets and 2-joins overlap in Berge graphs.

Lemma 12.5 (Trotignon and Vušković 2012) $0 \leq \alpha_C \leq \alpha_{AC}, \alpha_{BC} \leq \alpha_X \leq \alpha_{AC} + \alpha_{BC}$.

Proof. The inequalities $0 \leq \alpha_C \leq \alpha_{AC}, \alpha_{BC} \leq \alpha_X$ are trivially true. Let D be a maximum weighted stable set of $G[X_1]$. We have:

$$\alpha_X = w(D) = w(D \cap A_1) + w(D \cap (C_1 \cup B_1)) \leq \alpha_{AC} + \alpha_{BC}.$$

□

Lemma 12.6 (Trotignon and Vušković 2012) *If (X_1, X_2) is an odd 2-join of G, then $\alpha_C + \alpha_X \le \alpha_{AC} + \alpha_{BC}$.*

Proof. Let D be a stable set of $G[X_1]$ of weight α_X and C a stable set of $G[C_1]$ of weight α_C. In the bipartite graph $G[(C \cup D]$, we denote by Y_A (resp. Y_B) the set of those vertices of $C \cup D$ for which there exists a path in $G[C \cup D]$ joining them to some vertex of $D \cap A_1$ (resp. $D \cap B_1$). Note that from the definition, $D \cap A_1 \subseteq Y_A$, $D \cap B_1 \subseteq Y_B$ and there are no edges between $Y_A \cup Y_B$ and $(C \cup D) \setminus (Y_A \cup Y_B)$. We claim that $Y_A \cap Y_B = \emptyset$, and Y_A is anticomplete to Y_B. Suppose not. Then there exists a path P in $G[C \cup D]$ from a vertex of $D \cap A_1$ to a vertex of $D \cap B_1$. We may assume that P is minimal with respect to this property, and so the interior of P is in C_1; consequently P is of even length because $G[C \cup D]$ is bipartite. This contradicts the assumption that (X_1, X_2) is odd. Now we set:

- $Z_A = (D \cap Y_A) \cup (C \cap Y_B) \cup (C \setminus (Y_A \cup Y_B))$;
- $Z_B = (D \cap Y_B) \cup (C \cap Y_A) \cup (D \setminus (Y_A \cup Y_B)$.

From all the definitions and properties above, Z_A and Z_B are stable sets and $Z_A \subseteq A_1 \cup C_1$ and $Z_B \subseteq B_1 \cup C_1$. So, $\alpha_C + \alpha_X = w(Z_A) + w(Z_B) \le \alpha_{AC} + \alpha_{BC}$. □

Lemma 12.7 (Trotignon and Vušković 2012) *If (X_1, X_2) is an even 2-join of G, then $\alpha_{AC} + \alpha_{BC} \le \alpha_C + \alpha_X$.*

Proof. Let A be a stable set of $G[A_1 \cup C_1]$ of weight α_{AC} and B a stable set of $G[B_1 \cup C_1]$ of weight α_{BC}. In the bipartite graph $G[A \cup B]$, we denote by Y_A (resp. Y_B) the set of those vertices of $A \cup B$ for which there exists a path P in $G[A \cup B]$ joining them to a vertex of $A \cap A_1$ (resp. $B \cap B_1$). Note that from the definition, $A \cap A_1 \subseteq Y_A$, $B \cap B_1 \subseteq Y_B$, and $Y_A \cup Y_B$ is anticomplete to $(A \cup B) \setminus (Y_A \cup Y_B)$. We claim that $Y_A \cap Y_B = \emptyset$ and Y is anticomplete to Y_B. Suppose not, then there is a path P in $G[A \cup B]$ from a vertex of $A \cap A_1$ to a vertex of $B \cap B_1$. We may assume that P is minimal with respect to this property, and so the interior of P is in C_1; consequently it is of odd length because $G(A \cup B)$ is bipartite. This contradicts the assumption that (X_1, X_2) is even. Now we set:

- $Z_D = (A \cap Y_A) \cup (B \cap Y_B) \cup (A \setminus (Y_A \cup Y_B))$;
- $Z_C = (A \cap Y_B) \cup (B \cap Y_A) \cup (B \setminus (Y_A \cup Y_B))$.

From all the definitions and properties above, Z_D and Z_C are stable sets and $Z_D \subseteq X_1$ and $Z_C \subseteq C_1$. So, $\alpha_{AC} + \alpha_{BC} = w(Z_C) + w(Z_D) \leq \alpha_C + \alpha_X$. □

The two lemmas above allow to construct blocks of decomposition of a 2-join that preserve being Berge and allow to keep track of α (see [28] for the precise definition of the blocks). Interestingly, 2-joins are used to compute α in other classes of graphs (while they seem to be hard to use in general): in claw-free graphs (see Faenza, Oriolo and Stauffer [44]), and in even-hole-free graphs with no star cutsets (see Trotignon and Vušković [94]).

In [28], the inequalities above are used to prove the following. Many technicalities are needed, and some of them come from the fact that the blocks of decomposition for 2-joins that keep track of α do not preserve being balanced-skew-partition-free. Also, it is proved in [28] that for Berge graphs with no balanced skew partition, there exist *extreme* decompositions, that are decompositions such that one of the block is basic. These are very convenient for proofs by induction. To handle all these technicalities, it is convenient (if not mandatory) to work with trigraphs, but here we state the result for graphs.

Theorem 12.8 (Chudnovsky, Trotignon, Trunck and Vušković 2012) *There exists an $O(n^7)$ time algorithm whose input is a Berge graph with no balanced skew partition and whose output is a maximum weighted stable set of G and a colouring of G.*

So far, no one knows how skew partitions could be handled to provide a polynomial colouring algorithm. One might think that a Berge graph uniquely decomposable with a balanced skew partition must have an even pair. To support this idea, Chudnovsky and Seymour [25] studied the structure of Berge graphs with no K_4 and no even pair. They describe them quite precisely.

Theorem 12.9 (Chudnovsky and Seymour 2012) *If G is a 3-connected K_4-free Berge graph with no even pair, and with no clique cutset, then one of G, $\overline{G}$ is the line graph of a bipartite graph.*

This theorem was generalized by Zwols [98] to $\{K_4, \text{odd holes}\}$-free graphs (where non-perfect exceptions exist: $\overline{C_7}$, and a special graph on

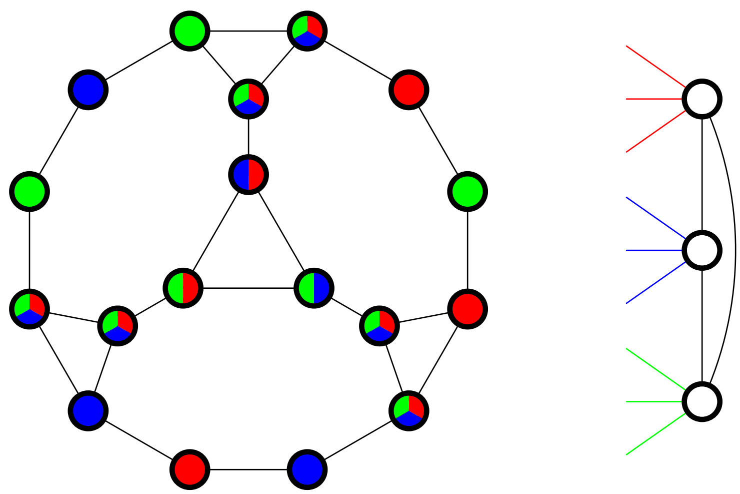

Figure 10: The WBGKSF (Worst Berge Graph Known So Far). Red (resp. green, blue) edges go to red (resp. green, blue) vertices.

eleven vertices). Unfortunately, it seems that this theorem does not generalize to larger values of ω, as shown by the WBGKSF, a graph G (discovered by Chudnovsky and Seymour, unpublished) represented on Fig. 10. Every edge in G is the middle edge of a P_4. This means that in the complement, every pair of non-adjacent vertices can be linked by a P_4. Therefore, $\overline{G}$ has no even pairs. However, G is perfect, and the balanced skew partition is the only outcome of Theorem 6.1 satisfied by G. In fact, G has even pairs, so the next conjecture (unpublished) could still be true. It is quite challenging, but it is not clear whether it would help to colour perfect graphs because even pairs in the complement do not seem usable.

Conjecture 12.10 (Thomas, 2002) *If a Berge graph G is uniquely decomposable by a balanced skew partitition, then one of G or $\overline{G}$ has an even pair.*

I feel the next two questions as the most important ones about perfect graphs.

Question 12.11 *Describe the structure of Berge graphs with no even pairs.*

Question 12.12 *Describe a combinatorial polynomial time algorithm that colours every Berge graphs.*

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